On Edge-Szeged & G/A Edge-Szeged Index of Standard Graphs

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Abstract: Wiener Index (see [6]) is the first topological index based on graph-distances. The next significant index is due to Gutman (see [3]) based on the nearity of vertices relative to the edges of the graph. Further, the Geometric/Arthimetic – mean index corresponding to the Wiener index (see [2]) is also considered. The present work is an analogue to edges.

Keywords: Szeged Index, G/A – Szeged index, edge Szeged index and G/A – edge Szeged index.

§1.**Introduction and Basic Results:**

Throughout this paper, by a graph we mean a non-empty, finite, simple and connected one. Chemical graphs are the graph-based descriptions of molecules where atoms are represented by vertices and bonds by edges. The association of a non-negative real number to a graph G is called a 'topological index' of G. These indices have significant applications to the graphs associated to the molecular structure of a chemical compound (designated as molecular graphs).

For the standard notation and results we refer Bondy & Murthy ([1]).

For ready reference, we give the following.

Definition 1.1([5]) : G, H are disjoint graphs. The Tensor product of G and H, denoted by G ^ H (that is isomorphic to H ^ G) is the graph whose vertex set is V(G) x V(H) and the edge set being the set of all edges of the form (u, v) (u^1, v^1) where u, $u^1 \in V(G)$, $v, v^1 \in V(H)$, $uu^1 \in E(G)$ and $vv^1 \in E(H)$.

Result 1.2 [5]: G_1, G_2 are connected graphs. Then $G_1 \wedge G_2$ is connected if and only if (iff) either G_1 or G_2 contains an odd cycle.

Result 1.3 [4]: For m, n ≥ 2, $K_m \wedge K_n$ (isomorphic to $K_n \wedge K_m$) is a simple, finite and (m – 1)(n – 1) – regular graph with mn vertices and ½ mn (m –1)(n – 1) edges. Further it is

(a) bipartite only when one of m, n is 2 **(b)** connected when atleast one of m, n is ≥ 3.

Result 1.4 [4]: $C_m \wedge C_n$ is a simple, 4 – regular graph with mn vertices and hence 2mn edges.

Result 1.5: For m, n ≥ 3, $C_m \wedge C_n$ is a bipartite graph iff atleast one of m, n is even.

To discuss about edge analogues, we first introduce the following:

Definition 1.6: Let G be a graph with edge set E(G). Then the edge Szeged Index of G, denoted by Ed- Sz(G) is defined to be

$$\sum_{e=uv\in E(G)} m_u(e/G)m_v(e/G) \text{ where}$$

$M_u(e/G) = \{f \in E(G) : d(u, f) < d(v, f)\}$,

$M_v(e/G) = \{f \in E(G) : d(v, f) < d(u, f)\}$;

$m_u(e) = |M_u(e)|$, $m_v(e) = |M_v(e)|$ ('| |' denotes the cardinality)

and

if f=xy, then

d(u, f) = Min{d(u, x), d(u, y)} and d(v, f) = Min{d(v, x), d(v, y)}.

Definition 1.7: Let G be a graph with edge set E(G). The Geometric/Arithmetic mean – edge Szeged Index of G, denoted by G/A – Ed Sz(G) is defined to be

$$G/A - Ed\ \ Sz(G) = \sum_{e=uv\in E(G)} \frac{\sqrt{m_u(e/G)m_v(e/G)}}{(m_u(e/G)+m_v(e/G))/2} = 2\sum_{e=uv\in E(G)} \frac{\sqrt{m_u(e/G)m_v(e/G)}}{m_u(e/G)+m_v(e/G)}.$$

Convention 1.8: When there is only one graph under consideration instead of (e/G) we write (e) only.

§2. **Results Related to Standard Graphs:**

Theorem 2.1: For the complete graph K_n ($n \geq 2$),

(i) $\quad Ed\text{-}Sz(K_n) = \dfrac{n(n-1)(n-2)^2}{2}$;

(ii) $\quad G/A - Ed\ Sz(K_n) = \dfrac{n(n-1)}{2}$.

Proof: Let $e = uv \in E(K_n)$.
By definition, $d(u, e) = 0 = d(v, e) \Rightarrow e \notin m_u(e) \cup m_v(e)$.
Let $f = uy \in E(K_n)$ with $y \neq v$. Now $d(u, f) = 0 < 1 = d(v, f) \Rightarrow u \in M_u(e)$.
Let f = xy where $x, y \notin \{u, v\}$. Now $d(u, f) = 1 = d(v, y) \Rightarrow f \notin m_u(e) \cup m_v(e)$.
$\Rightarrow m_u(e) = | M_u(e)| = d(u) - 1 = (n - 1) - 1 = (n - 2)$.
Similarly, $m_v(e) = (n-2)$.

This is true for all the edges of K_n. Since K_n has $\dfrac{n(n-1)}{2}$ *edges*

Follows that

$$Ed\text{-}Sz(K_n) = \sum_{e\in E(K_n)} (n-2)(n-2)$$

$$= |E(K_n)|(n-2)^2$$

$$= \frac{n(n-1)}{2}(n-2)^2.$$

Since $m_u(e) = m_v(e)$, follows that $2\dfrac{\sqrt{m_u(e)m_v(e)}}{m_u(e)+m_v(e)} = 1$.

Hence $G/A - Ed\ Sz(K_n) = \sum_{e\in E(K_n)} 1$

$$= |E(K_n)|$$

$$= \frac{n(n-1)}{2}.$$

Theorem 2.2: For the complete bipartite graph $K_{m,n}$ (m, n ≥ 1),

(i) Ed-Sz($K_{m, n}$) = (m – 1)(n – 1) mn ;

(ii) **G/A –** Ed Sz($K_{m, n}$) = $2\frac{\sqrt{(m-1)(n-1)}}{m+n-2}$ (whenever m+n ≥ 3).

(Observe that these are 0 when atlest one of m, n is 1)

Proof: Since $K_{1, 1} = K_2$ the first result is trivial when m = n = 1. We will not consider the 2nd one since the requirement is m+ n ≥ 3.

Let m+n ≥ 3 such that one of m, n is 1. Without loss of generality we can assume that m=1 and => n ≥ 2. Now any edge of $K_{1, n}$ is of the form uv_j (j = 1, …, n). Denote $e_j = uv_j$. Fix j.

Since d(u, e_j) = 0 = d(v, e_j) follows that $e_j \notin M_u(e_j) \cup M_{u_j}(e_j)$.

For $j_0 \in \{1, 2, …, n\} – \{j\}$,

d(u, e_{j_0}) = 0 = d(v_j, e_{j_0}) = min{1, 2} = 1 $\Rightarrow e_{j_0} \in M_u(e_j)$.

So follows that $m_u(e_j) = (n – 1)$ and $m_{v_1}(e_j) = 0$

⇨ $\sum_{e\in E(K_{1,n})} m_u(e_j)m_v(e_j) = 0.$

Hence Ed -$S_z(K_{1,n})$ = 0.

Further G/A – Ed $S_z(K_{1,n})$ = $2\sum_{e\in K_{1,n}} \frac{\sqrt{(n-1)0}}{(n-1)+0}$ = 0.

Now let m, n ≥ 2.

Let (X, Y) be a partition of of the vertex set of $K_{m, n}$, where X = {$u_1, u_2, …, u_m$} and Y = {$v_1, v_2, …,v_n$}.

Any edge of $K_{m, n}$ is of the form $e_{i, j} = u_i v_j$ (i= 1, 2,…, m; j = 1, 2,…, n). Since d(u_i, $e_{i,j}$) = 0=d(v_j, $e_{i,j}$) follows that $e_{i,j} \notin M_{u_i}(e_{i,j}) \cup M_{v_j}(e_{i,j})$............(2.2.1)

Fix (i,j).

Consider the edge e_{i,j_0} with $j_0 \in$ { 1, 2, …, n} – { j}.

Since $d(u_i, e_{i,j_0}) = 0$ *and* $d(v_j, e_{i,j_0})$ = Min{1,2} = 1

follows that $e_{i,j_0} \in M_{u_i}(e_{i,j})$ ………………………………(2.2.2)

Consider the edge $e_{i_0,j}$ *with* $i_0 \neq i$.

Since $d(u_i, e_{i_0,j})$ = Min{1,2} = 1 and

$d(v_j, e_{i_0,j}) = 0$ follows that $e_{i_0,j} \in M_{v_j}(e_{i,j})$ …………………..(2.2.3)

Consider the edge e_{i_0,j_0} *with* $i_0 \neq i$ and $j_0 \neq j$.

Now $d(u_i, e_{i_0,j_0}) = Min\{2,1\} = 1$ and $d(v_j, e_{i_0,j_0}) = Min\{1,2\} = 1$

follows that $e_{i_0,j_0} \notin M_{u_i}(e_{i,j}) \cup M_{v_j}(e_{i,j})$ ....(2.2.4)

From (2.2.1) - (2.2.4) follows that $m_{u_i}(e_{i,j}) = (n-1)$ *and* $m_{v_j}(e_{i,j}) = (m-1)$.

This is true for all $e_{i,j} \in E(K_{m,n})$.

So Ed – Sz($K_{m,n}$) = $\sum_{e=uv \in E(K_{m,n})} m_u(e) m_v(e)$

= (n – 1)(m – 1) |E($K_{m,n}$)|

= (m – 1) (n – 1) mn.

Now, G/A – Ed Sz($K_{m,n}$) = $\sum_{e \in E(K_{m,n})} \frac{2\sqrt{m_u(e) m_v(e)}}{m_u(e) + m_v(e)}$

$= 2\frac{\sqrt{(m-1)(n-1)}}{m+n-2} |E(K_{m,n})|$

$= 2\frac{\sqrt{(m-1)(n-1)}}{m+n-2} mn$.

Thus the proof is complete.

Theorem 2.3: For the path P_n ($n \geq 3$),

(i) Ed – Sz(P_n) = $\frac{(n-1)(n-2)(n-3)}{6}$;

(ii) G/A - Ed Sz(P_n) =

$\frac{2}{n-2}\sum_{i=1}^{n-3}\sqrt{i(n-2-i)}$ $\frac{2}{n-2}\sum_{i=1}^{n-3}\sqrt{i(n-2-i)}$ (with the convention, =0 when n=3).

($P_2 = K_2$ and is considered in Th.(2.1))

Proof: Let the vertex set of P_n be V(P_n) = {v_1, v_2, v_3, …, v_n}. The edges of P_n are $e_i = v_i v_{i+1}$ (i=1, 2, …, n-1). We observe that $M_{v_1}(e_1) = M_{v_n}(e_{n-1})$.

Further,

$M_{v_i}(e_i)$ = {e_1, …, e_{i-1}}, for i = 2, …, n-1 and $M_{v_{i+1}}(e_i)$ = { e_{i+1}, …, e_{n-1}} for i = 1, …, n-2

⇨ $m_{v_1}(e_1) = 0 = m_{v_n}(e_{n-1})$ …(2.3.1)

$m_{v_i}(e_i) = i - 1$ for i = 2, …, n-1 and

$m_{v_{i+1}}(e_i) = n - 1$ for i = 2, …, n-2 .

So, Ed – Sz(P_3) = $\sum_{i=1}^{2} m_{v_i}(e_i) m_{v_{i+1}}(e_i)$

= 0 + 0 = 0 (by (2.3.1))

$\Rightarrow$ G/A - Ed Sz(P_3) = 0.

For $n \geq 4$,

$$\text{Ed}-\text{Sz}(P_n) = \sum_{i=1}^{n-1} m_{v_i}(e_i) m_{v_{i+1}}(e_i)$$

$$= \sum_{i=2}^{n-2} (i-1)(n-1-i)$$

$$= \sum_{i=1}^{n-3} i(n-2-i) \text{ (Replacing i-1 by i)}$$

$$= (n-2)\sum_{i=1}^{n-3} i - \sum_{i=1}^{n-3} i^2$$

$$= \frac{(n-2)(n-3)(n-2)}{2} - \frac{(n-3)(n-2)(2n-5)}{6}$$

$$= \frac{(n-3)(n-2)}{6}[3(n-2)-(2n-5)]$$

$$= \frac{(n-1)(n-2)(n-3)}{6}.$$

$$\text{Ed}-\text{Sz}(P_n) = 2\sum_{i=1}^{n-1} \frac{\sqrt{m_{v_i}(e_i) m_{v_{i+1}}(e_i)}}{m_{v_i}(e_i) + m_{v_{i+1}}(e_i)}$$

$$= 2\sum_{i=2}^{n-2} \frac{\sqrt{(i-1)(n-1-i)}}{(i-1)+(n-1-i)}$$

$$= \frac{2}{n-2}\sum_{i=1}^{n-3} \sqrt{i(n-2-i)}.$$

This completes the proof of the theorem.

Theorem 2.4: For the cycle C_n ($n \geq 3$),

(i)
$$\text{Ed}-\text{Sz}(C_n) = \begin{cases} \left(\frac{n}{2}-1\right)^2 n & \text{if n is even,} \\ \left[\frac{n}{2}\right]^2 n & \text{if n is odd.} \end{cases}$$

(ii) G/A – Ed Sz(C_n) = n .

Proof : Let $n \geq 3$ and the vertex set of C_n, i.e $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.

Case(a): Let n be even and = 2m ($m \geq 2$).

The edges of C_{2m} are $e_i = v_i v_{i+1}$ ($i = 1, \ldots, 2m$) with the convention $v_{2m+1} = v_1$. Now

$$M_{v_i}(e_i)=\{e_{m+i+1},..,e_{2m+i-1}\}\Rightarrow m_{v_i}(e_i)=(m-1)$$

and

$$M_{v_{i+1}}(e_i)=\{e_{i+1},e_{i+2},...,e_{m+i-1}\}\Rightarrow m_{v_{i+1}}(e_i)=(m-1)$$

(The edges e_i and e_{m+i} are missing in the enumeration)
with the convention $e_k = e_{k-2m}$ for $2m+1 \leq k \leq 4m-1$.
So

$$Ed-Sz(C_n)=\sum_{i=1}^{2m} m_{v_i}(e_i).m_{v_{i+1}}(e_i)$$

$$=(m-1)^2 2m=\left(\frac{n}{2}-1\right)^2 n$$

$$Since\ m_{v_i}(e_i)=m_{v_{i+1}}(e_i)\ for\ all\ i\in\{1,2,...,2m\}, we\ have$$

$$G/A-Ed\ Sz(C_n)=\sum_{i=1}^{2m}1=2m=n.$$

Case(b): Let n be odd and = 2m + 1($m \geq 1$).

The edges of e_{2m+1} are $e_i = v_iv_{i+1}$ (i = 1, …, 2m+1) with the convention $v_{2m+2} = v_1$. Now

$$M_{v_i}(e_i)=\{e_{m+i+1},e_{m+i+2}...,e_{2m+i}\}\Rightarrow m_{v_i}(e_i)=m$$

and

$$M_{v_{i+1}}(e_i)=\{e_{i+1},e_{i+2},...,e_{m+i}\}\Rightarrow m_{v_{i+1}}(e_i)=m.$$

(The edge e_i is missing in the enumeration)
with the convention $e_k = e_{k-(2m+1)}$ for $2m+2 \leq k \leq 4m+1$.
So

$$Ed-Sz(C_{2m+1})=\sum_{i=1}^{2m+1} m_{v_i}(e_i)m_{v_{i+1}}(e_i)$$

$$= m^2(2m+1)=\left[\frac{n}{2}\right]^2 n.$$

As in case (**a**)

$$G/A-Ed\ Sz(C_{2m+1})=\sum_{i=1}^{2m+1}1=(2m+1)=n.$$

This completes the proof of the theorem.

Theorem 2.5: For the wheel K_1 V C_n ($n \geq 3$),

(i) Ed-Sz(K_1 V C_n) = $\left\{\left(\frac{n}{2}\right)^2+4n-10\right\}n$ when n is even,

$\left\{\left[\frac{n}{2}\right]^2+10\left[\frac{n}{2}\right]-5\right\}n$ when n is odd.

(ii) G/A- Ed Sz(K_1 V C_n) = $\begin{cases} \left\{1+\dfrac{2\sqrt{2}}{(2n-3)}\sqrt{2n-5}\right\}n \text{ when n is even,} \\ \left\{1+\dfrac{2\sqrt{2}\sqrt{4\left[\frac{n}{2}\right]-3}}{\left(4\left[\frac{n}{2}\right]-1\right)}\right\}n \text{ when n is odd.} \end{cases}$

Proof: Let $V(K_1) = \{u_0\}$ and $V(C_n) = \{v_1, v_2,\ldots,v_n\}$.
Now $E(K_1 \text{ V } C_n) = \{u_0v_i : i = 1, 2,\ldots, n\} \cup \{v_iv_{i+1} : i = 1,2,\ldots,n\}$
(with the convention $v_{n+1} = v_1$).
Case(i): Let n be even. So we can write n=2m ($m \geq 2$)
Denote $e_i = v_iv_{i+1}$ and $f_i = u_0v_i$ for $i = 1,2,\ldots,2m$.

Now, $M_{v_i}(e_i)=\{e_{m+i+1},\ldots,e_{2m+i-1}\}\cup\{f_i\}\Rightarrow m_{v_i}(e_i)=m,$

$M_{v_{i+1}}(e_i)=\{e_{i+1},e_{i+2}\ldots,e_{m+i-1}\}\cup\{f_{i+1}\}\Rightarrow m_{v_{i+1}}(e_i)=m.$

(with the convention $e_k = e_{k-2m}$ for $2m+1 \leq k \leq 4m-1$)
and

$M_{u_0}(f_i)=\{f_j : j\in N_{2m}-\{i\}\}\cup\{e_j : j\notin N_m-\{i-2,i-1,i,i+1\}\}$

$\Rightarrow M_{u_0}(f_i)$ = (2m-1) + (2m-4) = 4m-5

(with the convention $e_0 = e_{2m}$, $e_{-1} = e_{2m-1}$, $e_{-2} = e_{2m-2}$, $e_{2m+1} = e_1$)
and

$M_{v_i}(f_i)=\{e_i,e_{i-1}\}\Rightarrow M_{v_i}(f_i)=2.$

So

Ed-Sz(K_1 V C_n)= $\sum_{i=1}^{2m}m_{v_i}(e_i)m_{v_{i+1}}(e_i)+\sum_{i=1}^{2m}m_{v_0}(f_i)m_{v_i}(f_i)$

$= m^2(2m) + (4m-5)2(2m) = (m^2+8m-10)2m = \left(\left(\frac{n}{2}\right)^2+4n-10\right)n.$

G/A- Ed Sz(K_1 V C_n) = $\sum_{i=1}^{2m}1+\sum_{i=1}^{2m}2\dfrac{\sqrt{2(4m-5)}}{(2+4m-5)}$

$= 2m+\dfrac{2\sqrt{2}\sqrt{4m-5}}{4m-3}2m$

$=\left\{1+\dfrac{2\sqrt{2}}{2n-3}\sqrt{2n-5}\right\}n.$

Case(b): Let n be odd. So we can write n = 2m+1 ($m \geq 1$).
Denote $e_i = v_iv_{i+1}$ and $f_i = u_0v_i$ for $i = 1,2,\ldots,(2m+1)$.
As in case (b) of Th.(2.4) and proceeding as in case (a), we get that

$m_{v_i}(e_i)$ = m+1 = $m_{v_{i+1}}(e_i)$
(with the convention $e_k = e_{k-2m+1}$ for $2m+2 \leq k \leq 4m$)
and
$m_{u_0}(f_i)=(2m+1-1)+(2m+1-4)=4m-3$ and $m_{v_i}(f_i)$=2

(with the convention $e_0 = e_{2m+1}$, $e_{-1} = e_{2m}$, $e_{-2} = e_{2m-1}$, $e_{2m+2} = e_1$).
So

$$\begin{aligned} \text{Ed-Sz}(K_1 \vee C_{2m+1}) &= \sum_{i=1}^{2m+1} m_{v_i}(e_i)m_{v_{i+1}}(e_i) + \sum_{i=1}^{2m+1} m_{u_0}(f_i)m_{v_i}(f_i) \\ &= (m+1)^2(2m+1) + 2(4m-3)(2m+1) \\ &= (m^2 + 10m - 5)(2m+1) \\ &= \left\{\left[\frac{n}{2}\right]^2 + 10\left[\frac{n}{2}\right] - 5\right\} n \end{aligned}$$

$$\begin{aligned} \text{G/A- Ed Sz}(K_1 \vee C_{2n+1}) &= \sum_{i=1}^{2m+1} 1 + 2\sum_{i=1}^{2m+1} \frac{\sqrt{2(4m-3)}}{2+(4m-3)} \\ &= (2m+1) + \frac{2\sqrt{2(4m-3)}}{(4m-1)}(2m+1) \\ &= \left\{1 + \frac{2\sqrt{2(4m-3)}}{4m-1}\right\}(2m+1) \\ &= \left\{1 + \frac{2\sqrt{2}\sqrt{(4\left[\frac{n}{2}\right]-3}}{(4\left[\frac{n}{2}\right]-1)}\right\} n. \end{aligned}$$

This completes the proof of the theorem.

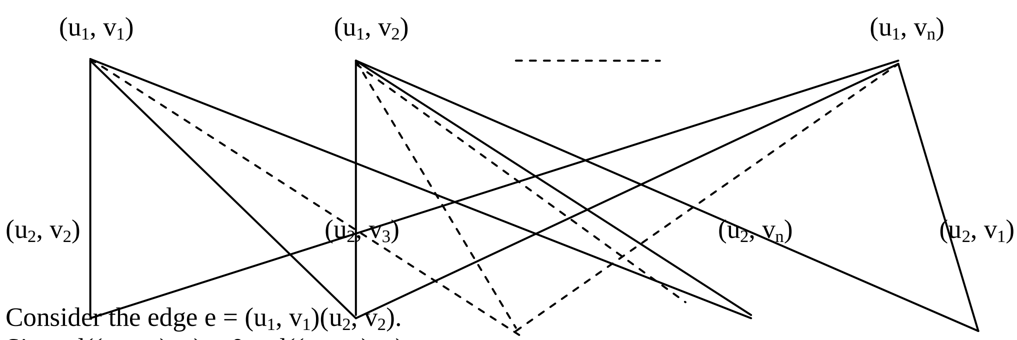
§3. Results Related to Tensor product of standard Graphs:

Theorem 3.1: For any integer $n \geq 3$,

i) $\text{Ed-Sz}(K_2 \wedge K_n) = 4n(n-1)(n-2)^2$,

ii) $\text{G/A – Ed Sz}(K_2 \wedge K_n) = n(n-1)$.

Proof: By Result (1.3), it follows that $K_2 \wedge K_n$ is connected and bipartite.
Let $V(K_2) = \{u_1, u_2\}$; $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Now, the vertex set of $K_2 \wedge K_n$ is $\{u_i, v_j\}$: i=1,2 ; j = 1, 2,…,n and the edge set $E(K_2 \wedge K_n)$ is the set of elements of the form $(u_1, v_j)(u_2, v_{j'})$, $1 \leq j \neq j' \leq n$.
A bipartition of $V(K_2 \wedge K_n)$ is {X, Y} where $X = \{(u_1, v_j) : j = 1, 2,\ldots, n\}$ and $Y = \{(u_2, v_j): j = 1,2,\ldots,n\}$.
A diagrametic representation of $K_2 \wedge K_n$ is the following.



Consider the edge $e = (u_1, v_1)(u_2, v_2)$.
Since $d((u_1, v_1), e) = 0 = d((u_2, v_2), e)$,
it follows that 'e' is not in $M_{(u_1,v_1)}(e)$ as well as in $M_{(u_2,v_2)}(e)$ ……. (3.1.1)
For j = 3,…,n , denote $e_j = (u_1, v_1)(u_2, v_j)$.

Since $d((u_1, v_1), e_j) = 0$ and $d((u_2, v_2), e_j) = \min\{1, 2\} = 1$, follows that $e_j \in M_{(u_1,v_1)}(e)$ *for* j=3,...,n ....(3.1.2)

For j = 3,…,n, denote $e_{j'} = (u_1\ v_2)(u_2\ v_j)$

$d((u_1,v_1),e_{j'}) = 1$ and $d((u_2,v_2),e_{j'}) = \min\{3.2\} = 2$,

follows that $e_{j'} \in M_{(u_1,v_1)}(e)$ for j= 3,...,n ....(3.1.3).

Since $d((u_1,v_1), (u_1,v_2)(u_2,v_1) = 2 = d((u_1,v_2), (u_1.v_2)(u_2,v_1)$, follows that $(u_1.v_2)(u_2,v_1)$ is not in $M_{(u_1,v_1)}(e)$ as well as in $M_{(u_2,v_2)}(e)$. ……..(3.1.4).

for j = 3,…,n,

since $d((u_1, v_1), (u_1, v_j)(u_2, v_2) = 1$ and $d((u_2,v_2), (u_1,v_j)(u_2,v_2)) = 0$,

Follows that $(u_1,v_j)(u_2,v_2) \in M_{(u_2,v_2)}(e)$ ....(3.1.5).

Since $d((u_1,v_1), (u_1,v_5)(u_2,v_1) = 3$ and $d((u_2, v_2), (u_1, v_j)(u_2,v_1)) = 1$

follows that $(u_1,v_j)(u_2,v_1) \in M_{(u_2,v_2)}(e)$ ....(3.1.6)

for j, j′ ∈ {3,...,n} and j′ ≠ j.

Since $d((u_1,v_1),(u_1,v_j)(u_2,v_{j'}) = 1 = d((u_2,v_2),(u_1,v_j)(u_2,v_{j'})$ follows that

this edge $(u_1,v_1)(u_2,v_{j'})$ is not in $M_{(u_1,v_1)}(e)$ as well as in $M_{(u_2,v_2)}(e)$ ……..(3.1.7).

From (3.1.1) – (3.1.7). it follows that $M_{(u_1,v_1)}(e) = 2(n-2) = M_{(u_2,v_2)}(e)$ ………(3.1.8).

Since $K_2 \wedge K_n$ is symmetric with respect to all the edges, it follows that for any edge of $K_2 \wedge K_n$, we get the same values as in (3.1.8).

Therefore, $\text{Ed-Sz}(K_2 \wedge K_n) = \sum_{e\in E(K_2 \Lambda K_n)} 2(n-2)2(n-2)$

$$= 4(n-2)^2 n(n-1)$$

$$= 4n(n-1)(n-2)^2.$$

Since $m_{(u_1,v_1)}(e) = m_{(u_2,v_2)}(e)$, it follows that

$\text{G/A – Ed Sz}(K_2 \wedge K_n) = |V(K_2 \Lambda K_n)| = n(n-1)$.

This completes the proof of the Theorem.

Remark 3.2: Observe that $K_2 \Lambda K_3 = C_6$. Now by Theorem (3.1), Ed-Sz($K_2 \Lambda K_3$) = $4.3(1)^2 = 24$ and by Theorem (2.4), $\text{Ed-Sz}(K_6) = \left(\frac{6}{2}-1\right)^2 .6 = 4.6 = 24$.

Theorem 3.3: For the integers m, n ≥ 3

(i) $\text{Ed-Sz}(K_m \wedge K_n) = 2mn\,(m-1)(n-1)\,[(m-1)(n-1)-1]^2$.

(ii) $\text{G/A-Sz}(K_m \wedge K_n) = \frac{1}{2}mn(m-1)(n-1)$.

Proof: Let $V(K_m) = \{u_1, u_2,\dots,u_m\}$ and $V(K_n) = \{v_1, v_2,\dots,v_n\}$.

Now $V(K_m \wedge K_n) = \{(u_i, v_j) : i = 1,2,\dots,m ; j = 1,2,\dots,n\}$.

$E(K_m \wedge K_n) = \left\{(u_i,v_j)(u_{i'},v_{j'}) : 1 \le i < i' \le m; 1 \le j < j' \le n\right\}$.

Clerarly $(K_m \wedge K_n)$ is a connected, m-bipatite graph with partition $\{X_1, \dots, X_m\}$, where $X_i = \{(u_i, v_j) : j=1,2,\dots,n\}$.

Further (**see Result(1.3)**) it is (m-1)(n-1)-regular with mn vertices and $\frac{1}{2}mn(m-1)(n-1)$ *edges* .

Since the graph is symmetric with regard to each edge, in the usual notation $m_{(u_i,v_j)}(e)$ and $m_{(u_{i'},v_{j'})}(e)$ are the same for all the edges 'e' of $K_m \wedge K_n$.
So, we calculate these for the edge e = $(u_1, v_1)(u_2, v_2)$.
As in **Theorem(3.1)**, it follows that

$$M_{(u_1,v_1)}(e) = \{(u_1,v_1)(u_2,v_j) : j = 3,...,n\} \cup$$
$$\{(u_1,v_1)(u_i,v_j) : i = 3,...,m;\ j = 2,...,n\} \cup$$
$$\{(u_1,v_2)(u_2,v_j) : j = 3,...,n\} \cup$$
$$\{(u_2,v_j)(u_i,v_2) : i = 3,...,m;\ j = 1,3,...,n\}$$
$$\Rightarrow m_{(u_1,v_1)}(e) = (n-2) + (m-2)(n-1) + (n-2) + (m-2)(n-1)$$
$$= 2(n-2) + 2(m-2)(n-1)$$
$$= 2[(m-1)(n-1) - 1] \quad \text{……………………………………..(3.3.1)}$$

and

$$M_{(u_2,v_2)}(e) = \{(u_1,v_j)(u_i,v_1) : i = 3,...,m,\ j = 2,3\} \cup$$
$$\{(u_1,v_3)(u_2,v_j) : j=1,2\} \cup$$
$$\{(u_1,v_j)(u_2,v_{j'}) : j = 4...,n,\ j' = 1,2\} \cup$$
$$\{(u_1,v_j)(u_i,v_1) : i = 3,...,m,\ j = 4,...,n\} \cup$$
$$\{(u_2,v_2)(u_i,v_j) : i = 3,...,m;\ j = 1,3,...,n\}$$

(with the convention that third and fourth sets are φ when n = 3).

$$\Rightarrow m_{(u_2,v_2)}(e) = 2(m-2)+2+2(n-3)+(m-2)(n-3)+(m-2)(n-1)$$
$$= 2[(m-1)(n-1)-1].$$

Now

$$\text{Ed-Sz}(K_m \wedge K_n) = \sum_{e\in E(K_m \wedge K_n)} m_{(u_1,v_1)}(e).m_{(u_2,v_2)}(e)$$
$$= 4\,[(m-1)(n-1) - 1]^2\,|E(K_m \wedge K_n)|$$
$$= 2\,mn\,(m-1)(n-1)\,[(m-1)(n-1) - 1]^2.$$

Since $m_{(u_1,v_1)}(e) = m_{(u_2,v_2)}(e)$, it follows that

$$G/A-Ed\ \text{Sz}(K_m \wedge K_n) = \sum_{e\in E(K_m \wedge K_n)} 1 = \frac{1}{2}mn(m-1)(n-1).$$

This completes the proof of the theorem.

Theorem 3.4: In the usual notation

(i) Ed-Sz($C_3 \wedge C_4$) = 2 (3) (4) $(3+4)^2$ = (24) (49) = 1176.

(ii) G/A-Ed Sz($C_3 \wedge C_4$) = 2 (3)(4) = 24.

Proof: Let $V(C_3) = \{u_1, u_2, u_3\}$ and $V(C_4) = \{v_1, v_2, v_3, v_4\}$. So $V(C_3 \wedge C_4) = \{(u_i, v_j) : i = 1,2,3 ; j = 1,2,3,4\}$. By **Results (1.4) &(1.5)** , ($C_3 \wedge C_4$) is a connected, bipartite,4-regular graph with 12 vertices and 24 edges. A bipartition of the vertex set of $C_3 \wedge C_4$ is {X,Y} where X = $\{(u_i, v_j) : i = 1,2\ 3;\ j = 1,3\}$ and Y = $\{(u_i, v_j) : i = 1,2,3;\ j = 2,4\}$. We observe that the graph is symmetric with regard to each edge. So, for any edge e = $(u, v)(u^1, v^1)$ of $C_3 \wedge C_4$, $m_{(u,v)}(e)$ and $m_{(u',v')}(e)$ are the same. Hence, we calculate these for e = $(u_1, v_1)(u_2, v_2)$. As in **Theorem (3.1)** ,

$$M_{(u_1,v_1)}(e) = \{(u_1,v_1)(u_3,v_2)\} \cup \{(u_1,v_1)(u_i,v_4): i = 2,3\} \cup \{(u_2,v_1)(u_3,v_j): j = 2,4\} \cup \{(u_2,v_3)(u_3,v_j): j = 2,4\}.$$

$\Rightarrow \quad m_{(u_1,v_1)}(e) = 1 + 2 + 2 + 2 = 7.$

$$M_{(u_2,v_2)}(e) = \{(u_1,v_3)(u_2,v_2)\} \cup \{(u_3,v_1)(u_1,v_4)\} \cup \{(u_3,v_1)(u_i,v_2): i = 1,2\} \cup \{(u_3,v_3)(u_i,v_2): i = 1,2\} \cup \{(u_3,v_3)(u_1,v_4)\}$$

$\Rightarrow m_{(u_2,v_2)}(e) = 1 + 1 + 2 + 2 + 1 = 7.$

Hence,

$$Ed - Sz(C_3 \Lambda C_4) = \sum_{e \in E(c_3 \Lambda c_4)} (7)(7)$$

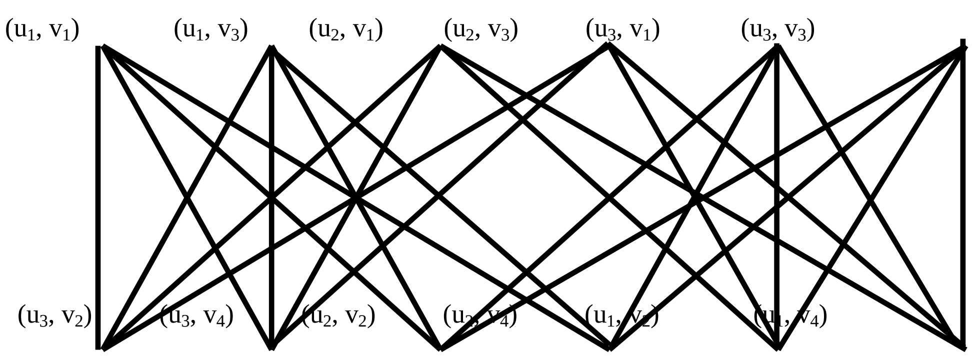
$$= (3+4)^2\,(2)(3)(4) = 1176.$$

Since, $m_{(u_1,v_1)}(e) = m_{(u_2,v_2)}(e)$, it follows that

$$G/A\text{-}Ed\ Sz(C_3 \Lambda C_4) = \sum_{e \in E(c_3 \Lambda c_4)} 1 = 2(3)(4) = 24.$$

This proves the theorem.

A diagrammatic representation of $C_3 \wedge C_4$ is



Now, we end this paper with the following:

In view of **Results (1.4) & (1.5)**, we have the following:

Open Problem 3.5: m, n ≥ 3 and one of m, n is odd, what are the values of Ed-Sz($C_m \wedge C_n$) and G/A – Ed Sz($C_m \wedge C_n$) ?

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